Renormalization and disorder : a simple toy model

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References

B. Derrida, M. Retaux
Journal of Statistical Physics 156, 268-290 (2014)

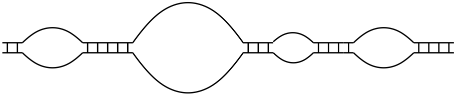
X. Chen, V. Dagard, B. Derrida, Z. Shi
Journal of Physics A 53, 195202 (2020)

B. Derrida, Z. Shi
Moscow Mathematical Journal 20, 695-709 (2020)

X. Chen, V. Dagard, B. Derrida, Y. Hu, M. Lifshits, Z. Shi
Annals of Probability 49, 637-670 (2021)

The Poland Scheraga model

Model of DNA denaturation



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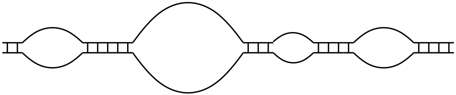
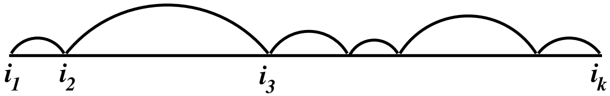


Model of depinning

The Poland Scheraga model (1966)



- the contact energy at position i is ϵ_i
- the weight of a loop of length n is

$$\omega(n) \sim \frac{s^n}{n^c}$$

$$Z_L = \sum_{k \geq 2} \sum_{1 < i_2 \cdots < i_{k-1} < L} \omega(i_2 - i_1) \cdots \omega(i_k - i_{k-1}) \exp\left[-\frac{\epsilon_{i_1} + \epsilon_{i_2} + \cdots \epsilon_{i_k}}{T}\right]$$

Note that one can always choose s=1

Main message

- Poland Scheraga + disorder

  A phase transition of the Berezinski Kosterlitz Thouless type

  Tang, Chaté 2001

  Monthus 2017

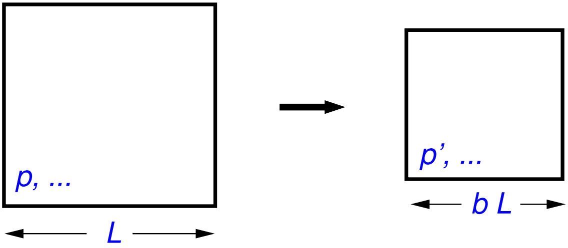
  Strong disorder ⇒ infinite order transition

- A renormalization equation

  D, Retaux 2014

$$\frac{\partial R(x,\tau)}{\partial \tau} = \frac{\partial R(x,\tau)}{\partial x} + \frac{1}{2}\int_0^x R(x_1,\tau)\, R(x-x_1,\tau)\, dx_1$$

The renormalization group



Renormalization transformation

$$\boxed{(p', \cdots) \;=\; \mathcal{R}_b\,(p, \cdots)}$$

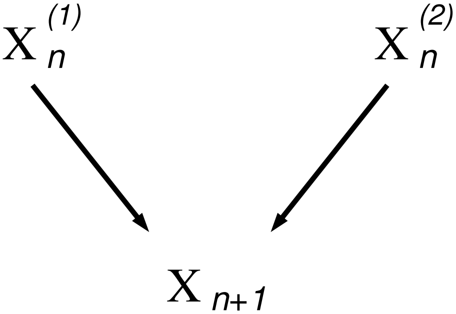
Look for the fixed point $\quad (p^*, \cdots) \;=\; \mathcal{R}_b\,(p^* \cdots)$

Linearize $\mathcal{R}_b$ near the fixed point $\quad (p', \cdots) \;=\; \mathcal{L}_b\,(p \cdots)$

$\Rightarrow$ Critical exponents and universality

The toy model

Collet, Glaser, Eckmann, Martin 1984



$$X_{n+1} = \max[\, X_n^{(1)} + X_n^{(2)} - 1 \;,\; 0 \,]$$

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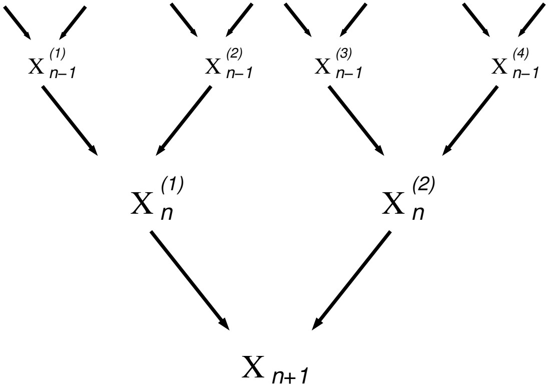

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Renormalization

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Exact renormalization (special case X_n are integers)

$$P_0(X) \quad \text{is given} \qquad ; \qquad P_n \quad \to \quad P_{n+1}$$

Renormalization

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$P_0(X)$ is given ; $P_n \rightarrow P_{n+1}$

$P_n(X)$ depends on an infinite number of parameters

Renormalization

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Exact renormalization (special case X_n are integers)

$P_0(X)$ is given ; $P_n \rightarrow P_{n+1}$

$P_n(X)$ depends on an infinite number of parameters

Define the generating function $H_n(z) = \sum_n P_n(X)\, z^X$

$$H_{n+1}(z) = \frac{H_n(z)^2 - H_n(0)^2}{z} + H_n(0)^2$$

A few facts

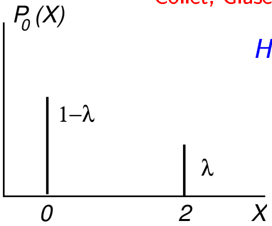
$$X_{n+1} = \max[\, X_n^{(1)} + X_n^{(2)} - 1 \,,\ 0\,]$$

$$\boxed{H_{n+1}(z) = \frac{H_n(z)^2 - H_n(0)^2}{z} + H_n(0)^2}$$

1. A phase transition

Collet, Glaser, Eckmann, Martin 1984

For example



$$H_0(z) = 1 - \lambda + \lambda z^2$$

$$\Rightarrow \quad \boxed{\lambda_c = \frac{1}{5}}$$

A few facts

$$X_{n+1} = \max[\, X_n^{(1)} + X_n^{(2)} - 1\ ,\ 0\,]$$

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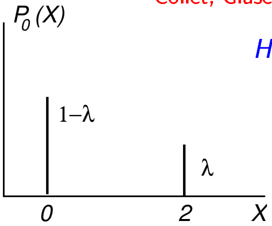

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2. A one parameter family of fixed points

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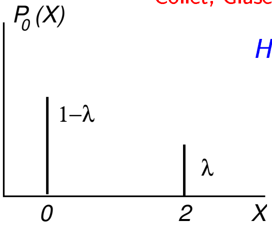

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2. A one parameter family of fixed points

3. None of them is accessible ($P^*(X)$ is not positive)

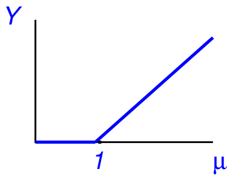
4. A phase transition of the Berezinski Kosterlitz Thouless type

Main question

$$X_{n+1} = \max[\, X_n^{(1)} + X_n^{(2)} - 1 \;,\; 0\,]$$

What is the limit Y *of* $\frac{X_n}{2^n}$

$$P_0(X) = \delta_{X,\mu}$$



Pure case

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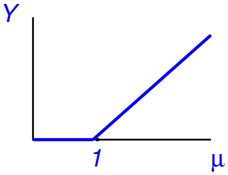
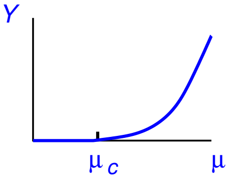


Pure case

$$P_0(X) = (1-\lambda)\,\delta_{X,0} + \lambda\,\delta_{X,\mu}$$



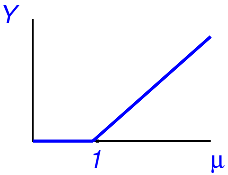
Disordered case

Main question

$$X_{n+1} = \max[\, X_n^{(1)} + X_n^{(2)} - 1 \,,\ 0\,]$$

What is the limit Y *of* $\dfrac{X_n}{2^n}$

$P_0(x) = \delta_{x-\mu}$



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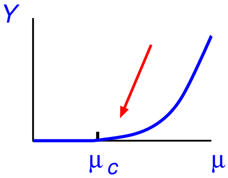


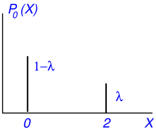
Y plays the role of the free energy

The critical behavior

$$X_{n+1} = \max[\, X_n^{(1)} + X_n^{(2)} - 1 \,,\; 0\,] \qquad H_n(z) = \sum_X P_n(X) z^X$$

A phase transition given by $\boxed{2H'(2) - H(2) = 0}$

Collet, Glaser, Eckmann, Martin 1984



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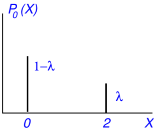

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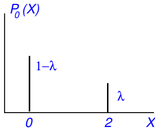

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$$\boxed{2H'(2) - H(2) \equiv \lambda - \lambda_c}$$

$$\lambda \leq \lambda_c$$

$$\lim_{n\to\infty} \frac{\langle X_n \rangle}{2^n} \to 0$$

The critical behavior

$$X_{n+1} = \max[\, X_n^{(1)} + X_n^{(2)} - 1 \;,\; 0\,] \qquad H_n(z) = \sum_X P_n(X) z^X$$

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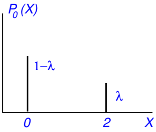

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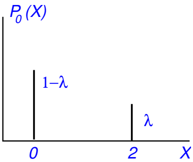
$\lambda \leq \lambda_c$	$\lambda > \lambda_c$
$\lim_{n\to\infty} \frac{\langle X_n \rangle}{2^n} \to 0$	$\boxed{\lim_{n\to\infty} \frac{\langle X_n \rangle}{2^n} \simeq \exp\left[-\frac{A}{\sqrt{\lambda - \lambda_c}} \right]}$

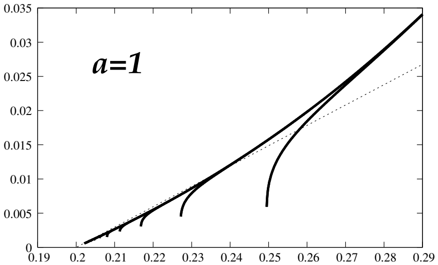
D., Retaux 2014

An essential singularity ?

$$\lim_{n\to\infty} \frac{\langle X_n \rangle}{2^n} \simeq \exp\left[-\frac{A}{\sqrt{\lambda-\lambda_c}}\right] \quad \Longleftrightarrow \quad \left(\log \frac{\langle X_n \rangle}{2^n}\right)^{-2} \propto (\lambda - \lambda_c)$$



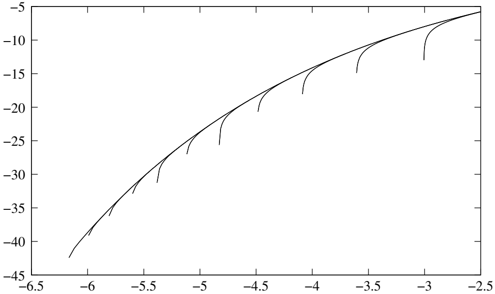
$$\lambda_c = \frac{1}{5}$$



$$\left(\log \frac{\langle X_n \rangle}{2^n}\right)^{-2} \quad \text{versus} \quad \lambda$$

A power law singularity ?

$$\frac{\langle X_n \rangle}{2^n} \propto (\lambda - \lambda_c)^\gamma \quad \Leftrightarrow \quad \log \frac{\langle X_n \rangle}{2^n} \sim \gamma \log(\lambda - \lambda_c)$$



$$\log \frac{\langle X_n \rangle}{2^n} \quad \text{versus} \quad \log(\lambda - \lambda_c)$$

What is proved

Chen Dagard D. Hu Lifshits Shi 2018

Initial distribution $P_0(X) = (1-\lambda)\delta_X + \lambda Q(X)$

$$\text{Particular case:} \quad Q(X) \sim \frac{C}{X^\alpha \, 2^X}$$

- If $Q(X)$ decays fast enough $(\alpha > 4)$ then $\lambda_c > 0$ and

$$\lim_n \frac{\langle X_n \rangle}{2^n} = \exp\left(-\frac{1}{(\lambda - \lambda_c)^{\frac{1}{2}+o(1)}} \right)$$

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- If $2 < \alpha < 4$ then $\lambda_c > 0$

$$\lim_n \frac{\langle X_n\rangle}{2^n} = \exp\left(-\frac{1}{(\lambda-\lambda_c)^{\nu+o(1)}}\right) \qquad \text{with} \quad \nu = \frac{1}{\alpha-2}$$

- If $\alpha \leq 2$ then $\lambda_c = 0$ (because $H'(2) = \infty$)

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A ?? special ?? family of initial conditions

D., Retaux 2014

$$P_n(X) = 2^{-X}\ \epsilon^2\ R(\epsilon X, \epsilon n) \qquad \textit{for} \qquad X > 0$$

and

$$P_n(0) = 1 - \sum_{X \geq 1} P_n(X)$$

A ?? special ?? family of initial conditions

D., Retaux 2014

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and

$P_n(0) = 1 - \sum_{X \geq 1} P_n(X)$

then one can show that for λ small

$$\frac{\partial R(x,\tau)}{\partial \tau} = \frac{\partial R(x,\tau)}{\partial x} + \frac{1}{2}\int_0^x R(x_1,\tau)\ R(x-x_1,\tau)\ dx_1$$

Still a difficult problem

Criticality

$$\int_0 R(x,\tau)\, x dx = 1$$

$$\frac{\partial R(x,\tau)}{\partial \tau} = \frac{\partial R(x,\tau)}{\partial x} + \frac{1}{2}\int_0^x R(x_1,\tau)\; R(x-x_1,\tau)\; dx_1$$

For

$$R(x,\tau) = A(\tau)\exp[-B(\tau)x]$$

one gets

$$\frac{dA(\tau)}{d\tau} = -B(\tau)A(\tau) \quad ; \quad \frac{dB(\tau)}{d\tau} = -\frac{A(\tau)}{2}$$

Kosterlitz Thouless renormalization

$$\frac{\partial R(x,\tau)}{\partial \tau} = \frac{\partial R(x,\tau)}{\partial x} + \frac{1}{2}\int_0^x R(x_1,\tau)\; R(x - x_1,\tau)\; dx_1$$

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Kosterlitz Thouless renormalization

$$R(x,\tau) = 4\frac{k^2}{\sin(k(\tau+\tau_0))^2}\; \exp\left[-\frac{2k\; x}{\tan(k(\tau+\tau_0))}\right]$$

($k \to 0$ is the critical case)

Other solutions

$$\frac{\partial R(x,\tau)}{\partial \tau} = \frac{\partial R(x,\tau)}{\partial x} + \frac{1}{2}\int_0^x R(x_1,\tau)\; R(x-x_1,\tau)\; dx_1$$

Physical solutions

$$R = \sum_{i=1}^{n} A_i(\tau)\, e^{-B_i(\tau) x}$$

with

$$\frac{dB_i}{d\tau} = -\frac{A_i}{2} \quad ; \quad \frac{dA_i}{d\tau} = -B_i A_i - \sum_{j\neq i} \frac{A_i A_j}{B_i - B_j}$$

Unphysical fixed solutions

$$R = \frac{4}{\tau^2} e^{-\frac{3x}{\tau}} \left[3\cos\left(\frac{\sqrt{3}x}{\tau}\right) + \sqrt{3}\sin\left(\frac{\sqrt{3}x}{\tau}\right)\right]$$

Scaling solutions

$$\frac{\partial R(x,\tau)}{\partial \tau} = \frac{\partial R(x,\tau)}{\partial x} + \frac{1}{2}\int_0^x R(x_1,\tau)\ R(x-x_1,\tau)\ dx_1$$

Scaling solutions along the critical manifold ($\int R(X)\ X\ dX = 1$)

$$R(x,\tau) = \frac{1}{\tau^2}\,\mathcal{R}\left(\frac{x}{\tau}\right)$$

- $\mathcal{R}(z) = 4\,e^{-2z}$
- Other scaling solutions

$\mathcal{R}(x) \sim x^{-\alpha}$ for $x \to \infty$

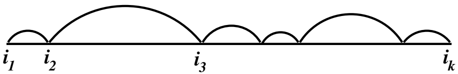
Laplace tranform: $\widetilde{\mathcal{R}}(p) = \int_0^\infty \mathcal{R}(z)\,e^{-pz}\,dz$

$$\widetilde{\mathcal{R}}(p) = -1 - p - p\,\frac{y'(p/2)}{y(p/2)}$$

where y is a Bessel function solution of

$$p^2 y'' + p y' - \left(p^2 + \left(\frac{\alpha-1}{2}\right)^2\right) y = 0$$

Phase transition in the Poland Scheraga model



$$\omega(n) \sim \frac{1}{n^c}$$

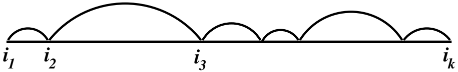
$$Z_L = \sum_{k\geq 2} \sum_{1<i_2\cdots<i_{k-1}<L} \omega(i_2 - i_1)\cdots\omega(i_k - i_{k-1}) \exp\left[-\frac{\epsilon_{i_1} + \epsilon_{i_2} + \cdots \epsilon_{i_k}}{T}\right]$$

The free energy $F_L = \log Z_L$. In the thermodynamic limit

$$f_\infty = \lim_{L\to\infty} \frac{F_L}{L}$$

- $T > T_c$ $\quad f_\infty = 0$ the unpinned phase
- $T < T_c$ $\quad f_\infty > 0$ the pinned phase

Phase transition in the pure case $\epsilon_i = \epsilon$



$$\omega(n) \sim \frac{1}{n^c}$$

- T_c is known

$$\exp\left[-\frac{\epsilon}{T_c}\right] = \sum_{n\geq 1} \omega(n)$$

- For $c > 2$ the transition is first order

$$f_\infty \sim (T_c - T)$$

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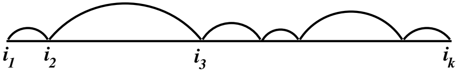

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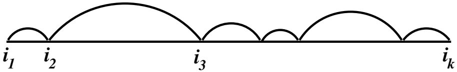
$$f_\infty \sim (T_c - T)$$

- For $1 < c < 2$ the transition is second order

$$f_\infty \sim (T_c - T)^{\frac{1}{c-1}}$$

Phase transition in the pure case $\epsilon_i = \epsilon$



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$$f_\infty \sim (T_c - T)$$

- For $1 < c < 2$ the transition is second order

$$f_\infty \sim (T_c - T)^{\frac{1}{c-1}}$$

- For $c < 1$ no transition

A short list of results in the disordered case

Alexander, Giacomin, Lacoin, Toninelli 2008 $\rightarrow$

- Irrelevant case ($1 < c < \frac{3}{2}$) and weak disorder :

$$T_c^{\text{quenched}} = T_c^{\text{annealed}}$$

same critical exponent as the pure case

- Marginal and relevant cases ($\frac{3}{2} \leq c$):
  $T_c^{\text{quenched}} \neq T_c^{\text{annealed}}$ and bounds on the difference

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$$T_c^{\text{quenched}} = T_c^{\text{annealed}}$$

same critical exponent as the pure case

- Marginal and relevant cases ($\frac{3}{2} \leq c$):
  $T_c^{\text{quenched}} \neq T_c^{\text{annealed}}$ and bounds on the difference

Giacomin, Toninelli 2006

The transition is always smooth

A short list of results in the disordered case

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Tang, Chaté 2001

Strong disorder ⇒ infinite order transition

D, Hakim, Vannimenus 1992

In the marginal case, $T_c^{\text{quenched}} - T_c^{\text{annealed}}$ is exponentially small

The hierarchical lattice

- $L = 2^n$
- all loops have lengths 2^k with $k = 1, 2, 3, \cdots$
- $$\boxed{Z_1^{(i)} = \exp\left[-\frac{\epsilon_i}{T}\right]}$$

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$$F_L = \log Z_L \quad ; \quad f_\infty = \lim_{L \to \infty} \frac{F_L}{L}$$

$f_\infty > 0$ is the pinned phase ; $f_\infty = 0$ is the unpinned phase

The hierarchical lattice in the pure case (for $1 < b < 2$)

Second order transition :

$$f_\infty \sim (T_c - T)^{\frac{1}{c-1}} \qquad \text{with} \quad c = 2 - \frac{\log b}{\log 2}$$

The hierarchical lattice with disorder

- Effect of a weak disorder:
  Harris criterion (1974) $\Rightarrow$ disorder is relevant if $c > \frac{3}{2}$
- For $c = \frac{3}{2}$ disorder is marginal relevant

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- Can one locate the transition?
- What is the nature of the transition?

The hierarchical lattice ($X_n = \log Z_{2^n}$)

$$X_{n+1} = G\left(X_n^{(1)} + X_n^{(2)}\right) \quad \Leftarrow \quad Z_{2L} = \frac{Z_L^{(1)} Z_L^{(2)} + b - 1}{b}$$

with

$$G(X) = X + \log\left(\frac{1 + (b-1)e^{-X}}{b}\right)$$

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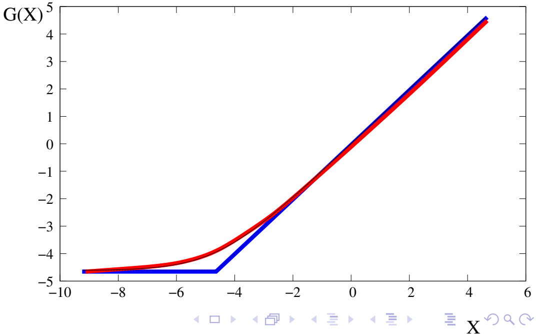
The toy model

$$X_{n+1} = G\left(X_n^{(1)} + X_n^{(2)}\right)$$

with

$$G(X) = \max(X, -a)$$



Conclusion

- Analysis of

$$\frac{\partial R(x,\tau)}{\partial \tau} = \frac{\partial R(x,\tau)}{\partial x} + \frac{1}{2}\int_0^x R(x_1,\tau)\ R(x-x_1,\tau)\ dx_1$$

- Going back to the hierarchical model
- Going back to the Poland Scheraga model