Decadence and Fall of the Academies

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31 May 2015

Abstract

The paper examines the structure of a deterministic model that describes the evolution of "academies". In particular, bright and average academy members are distinguished, and rules for membership replacement (a co-optation process) are established that include a bias for average members. Under these conditions, when the bright and the average members do not cooperate, the number of bright ones disappears exponentially. But if an expelling rule is added to the model, a critical state is reached that allows one to conclude on the fall or rise of the academies.

1 Introduction. The Co-optation Process

Here an "*academy*" is a generic group of people (the members) where new, entering members are chosen (co-opted) by the old ones. Then an academy is eventually a learned society, but also a political party, a medieval guild, the bureaucratic elite of an empire, or the group of partners of a counseling firm (and so on).

If $N(t)$ is the number of members at time t, we can assume that only N_A members are "*bright*" while the others N_B $(= N - N_A)$ are "*average*" or poor. And we shall assume that $\mu N(t)$ members $(0 < \mu < 1)$ leave spontaneously (or by natural causes) the group at time t; consequently $\alpha N(t)$ new members has to be co-opted by the remaining ones; obviously α is $\geq \mu$, if not the academy will evaporate.

Under a strong competitive pressure (or if, for example, the members survival is at a stake) it is reasonable to assume that *all* members try to co-opt new bright ones, but sometimes they do some mistake and select average ones; in this case we have only $(\alpha - \delta_A)\, N_A(t)$ and $(\alpha - \delta_B)\, N_B(t)$ new bright members, selected, respectively, by the bright and average ones. The new average members will be $\delta_A N_A(t)$ and $\delta_B N_B(t)$, at time t (Obviously the coefficient δ_A and δ_B are > 0 and $< \alpha$). We shall denote this scenario *altruistic*, see the Section 2.

On the contrary, in the absence of competition (isolation, protective environment, monopoly, exclusive access to natural resources) we can assume that the B-members, in the worst case, (almost) always co-opt average members ($\alpha N_B(t)$, at time t) while the A-members generally chose new bright members, but, again, they sometimes do some mistake and select average ones; consequently we have only $(\alpha - \delta_A)\, N_A(t)$ new bright members and $\delta_A N_A(t)$ new average ones, at time t. We shall denote this scenario *individualistic*, see the Sections 3 and 4.

In any case, the academy's dynamics can be described by a couple of differential equations (a "*linear plane autonomous system*")

$$\begin{aligned} N'_A(t) &= a.N_A(t) + b.N_B(t) \\ N'_B(t) &= c.N_A(t) + d.N_B(t) \end{aligned} \tag{1}$$

and $N'(t) = (a+c)\, N_A(t) + (b+d)\, N_B(t)$. If $N(t) = \mathit{const}$ and N_A, N_B are not constants then $a+c = b+d = 0$ and $\det \begin{pmatrix} a & b \\ c & d \end{pmatrix} = a\,(-b) - b\,(-a) = 0$; the system is *singular* (see Sansone and Conti [2], Chapter 2) and there is eventually a *critical point* $\left(\frac{b}{b-a}, \frac{-a}{b-a}\right)$ at the intersection of the lines

$$X + Y = 1, \quad Y = -\frac{a}{b} X. \tag{2}$$

Here $X = N_A/N$, $Y = N_B/N$, $0 \le X, Y \le 1$ and a/b has to be ≤ 0. The trace of the system's matrix is equal to $a - b$ so the critical point can *attract* (if $a < b$) or *repel* (if $a > b$) the solutions of the equations.

2 The Altruistic Scenario

In this case, the academy's dynamics is described by the differential equations

$$\begin{aligned} N'_A(t) &= (\alpha - \delta_A - \mu)\, N_A(t) + (\alpha - \delta_B)\, N_B(t) \\ N'_B(t) &= \delta_A N_A(t) + (\delta_B - \mu)\, N_B(t) \end{aligned} \tag{3}$$

and $N'(t) = (\alpha - \mu)\, N(t)$. If $N(t) = \mathit{const}$ ($\alpha = \mu$) the system is singular (end of Section 1) and $\mathrm{tr} \begin{pmatrix} -\delta_A\ , & \alpha - \delta_B \\ \delta_A\ , & \delta_B - \alpha \end{pmatrix} = -\,(\alpha + \delta_A - \delta_B) := -z$, where z is always > 0; the solutions converge towards the critical point

$$N \left(\frac{\alpha - \delta_B}{\alpha + \delta_A - \delta_B}, \frac{\delta_A}{\alpha + \delta_A - \delta_B} \right) \tag{4}$$

which is, generally, quite near to $(N, 0)$ [Figure 1].

In fact we have:

$$\frac{N_A(t)}{N} = \frac{\alpha - \delta_B}{z} + \left(\frac{N_A(0)}{N} - \frac{\alpha - \delta_B}{z} \right) e^{-zt} \tag{5}$$

so for $t \to \infty$, we arrive, exponentially and independently of the starting point, at a mixture of $\frac{(\alpha - \delta_B)N}{\alpha + \delta_A - \delta_B}$ A-members and of $\frac{\delta_A N}{\alpha + \delta_A - \delta_B}$ B-members, an obvious result of the altruistic cooperation (but a fraction of average members inevitably survive).

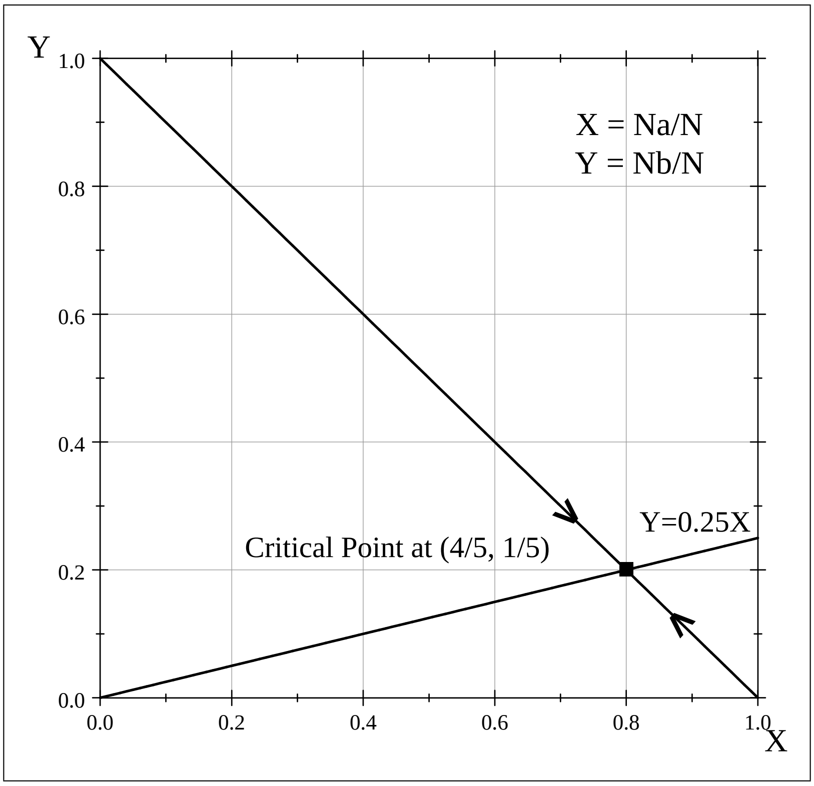

Figure 1 - The Altruistic Case. $\alpha = \mu = 0.10$, $\delta_A = 0.02$, $\delta_B = 0.02$, $z = 0.10$

3 The Individualistic Scenario

We shall consider an "extreme" individualistic behavior of the B-members, which *always* co-opt B-members; therefore:

$$\begin{aligned} N'_A(t) &= (\alpha - \delta_A - \mu)\, N_A(t) \\ N'_B(t) &= \delta_A N_A(t) + (\alpha - \mu)\, N_B(t) \end{aligned} \tag{6}$$

and again $N'(t) = (\alpha - \mu)\, N(t)$. If $N(t) = const$ $(\alpha = \mu)$, the bright members exponentially disappear $(N_A(t) = N_A(0)e^{-\delta_A t})$ and, anyway, when $\alpha > \mu$, the bright/average ratio goes rapidly to zero:

$$\frac{N_A(t)}{N_B\,(t)} = \frac{N_A(0).e^{-\delta_A t}}{N_B(0) + N_A(0)\,(1 - e^{-\delta_A t})}. \tag{7}$$

It is possible to devise some modification of the co-optation procedures (co-optation committees, "objectives" evaluations of the members performances, even random choices) but, in any case, we can only alter the value of δ_A (which is always > 0) and the B-members will rapidly predominate; the academy (isolated, protected or irrelevant) can nevertheless survive. A paradigmatic example, more or less fictional, is represented by the "grand *Academy of Lagado*", Swift [3] Part III, Chapter 5.

4 The Expelling Rule

However, to *contrast* the individualistic behavior (and to *improve* the altruistic one), a quite effective (but not so popular) approach is possible: we can prescribe that a certain number of members, $\nu N(t)$ (ν is a *fixed* parameter, $0 < \nu < 1$), *must* be expelled from the academy, at time t. Now it is reasonably to assume that A-members almost always will expel the average ones (if someone is left) while B-members generally will dismiss the bright ones. Here we shall consider the "worst" case, when the average *always* dismiss the bright:

$$\begin{aligned} N'_A(t) &= (\alpha - \delta_A - \mu)\, N_A(t) - \nu N_B(t) \\ N'_B(t) &= (\delta_A - \nu)\, N_A(t) + (\alpha - \mu)\, N_B(t) \end{aligned} \tag{8}$$

and $N'(t) = (\alpha - \mu - \nu)\, N(t)$. In the above equations $0 \le t < T_A(T_B)$ where $T_A(T_B)$ is the first zero of $N_A(N_B)$.

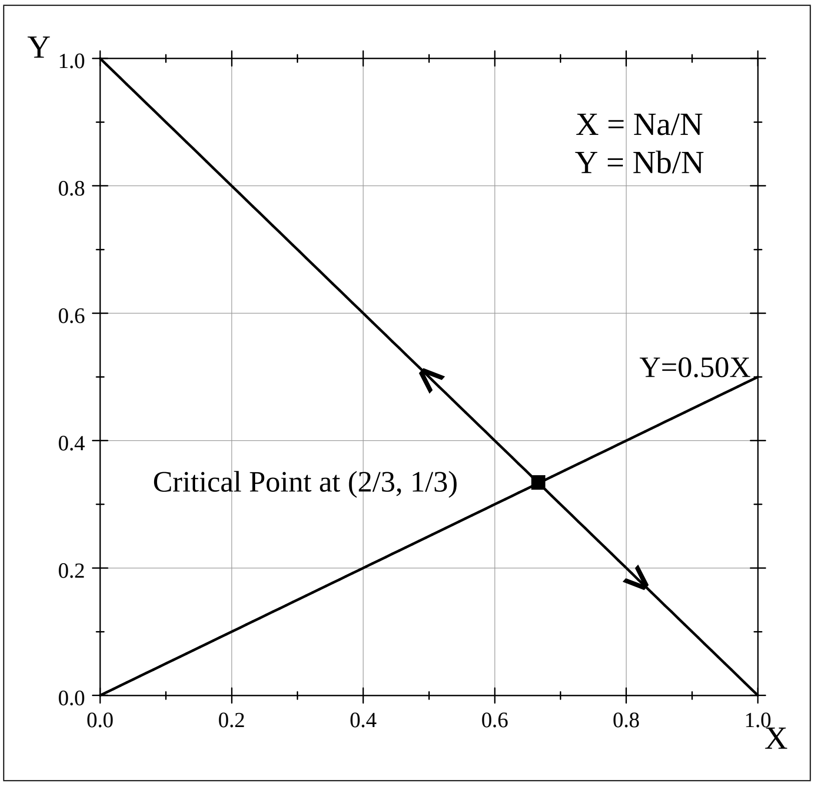

Figure 2 - The Expelling Rule. $\alpha = \mu + \nu = 0.16$, $\delta_A = 0.03$, $\nu = 0.06$, $z = 0.09$

Now when $N(t) = const$, $\alpha = \mu + \nu$ and $\operatorname{tr} \begin{pmatrix} \nu - \delta_A\ , & -\nu \\ \delta_A - \nu\ , & \nu \end{pmatrix} = 2\nu - \delta_A := z$, but if $\nu < \delta_A$ there is no critical point and $N_A(t)$ goes exponentially to zero, when $t \to \infty$.

Then we shall only consider the case $\nu > \delta_A$ (the expulsions compensate the wrong evaluations) where

$$N\left(\frac{\nu}{2\nu - \delta_A}, \frac{\nu - \delta_A}{2\nu - \delta_A}\right) \tag{9}$$

is a critical point (which *depend only* on δ_A and ν) [Figure 2] and

$$\frac{N_A(t)}{N} = \frac{\nu}{z} + \left(\frac{N_A(0)}{N} - \frac{\nu}{z}\right) e^{zt}. \tag{10}$$

If $\frac{N_A(0)}{N} > \frac{\nu}{2\nu-\delta_A}$ $(< \frac{\nu}{2\nu-\delta_A})$, N_A increase (decrease); if $\frac{N_A(0)}{N} = \frac{\nu}{2\nu-\delta_A}$, N_A and N_B are constants (they don't leave the critical point). Therefore if, at some time, $\frac{N_A}{N}\frac{2\nu-\delta_A}{\nu} < 1$ the academy will "*decade and fall*", inexorably; otherwise the academy will "*rise*".

Obviously the expulsions practice can also improve the altruistic scenario: for an "extreme" altruistic behavior, where *all the members* dismiss the average ones ($N_B'(t) = \delta_A N_A(t) + (\delta_B - \mu)\, N_B(t) - \nu N(t)$) and for $\nu < \delta_A$ (a small expulsion rate), the critical point $N\left(\frac{\alpha-\delta_B}{\mu-\delta_B+\delta_A}, \frac{\delta_A-\nu}{\mu-\delta_B+\delta_A}\right)$ is shifted nearer to $(N, 0)$. If $\nu = \delta_A$ the B-members exponentially disappear: $\frac{N_A(t)}{N} = 1 + \left(\frac{N_A(0)}{N} - 1\right) e^{-(\alpha-\delta_B)t}$.

5 Observations

In presence of an expelling rule, in the individualistic scenario, after some time N_A or N_B (if they are not constant) go to zero and the expulsions are not longer practicable. Then we can, for example, consider a non constant expulsion rate, decreasing when the academy became more "homogeneous" (almost all members are bright or average). In this case the problem is non linear, Lotka-Volterra like (see, for example, Murray [1]), N_A+ N_B cannot be constant. Anyway the solutions generally stabilize, at infinity, around the "all bright" or the "all average" solutions.

Perhaps more interesting is the case where there are many degrees of brightness, for example the bright/average/poor case, or the case of a continuously variable brightness (which leads to an integro-differential equation. For a "three degrees case" $(A/B/C)$, in the individualistic scenario, if the expulsion rate is enough high, there is always (on the plane $N_A + N_B + N_C = N$) a critical point, but now the solutions generally stabilize, at infinity, around a mixture of A/B or B/C or C/A members.

References

[1] Murray J D 1993. *Mathematical Biology* (Berlin: Springer-Verlag)

[2] Sansone G, Conti R 1956. *Equazioni differenziali non lineari* (Roma: Edizioni Cremonese) [1964. *Nonlinear Differential Equations* (Oxford: Pergamon Press)]

[3] Swift J 1985. *Gulliver's Travel* (London: Penguin Books, Penguin Classics)